

holds, and this forces  $n < \dim N(\mathbf{A}^{n+1})$ , which is impossible. A similar argument proves equality exists somewhere in the range chain. ■

**Property 2.** *Once equality is attained, it is maintained throughout the rest of both chains in (5.10.2). In other words,*

$$\begin{aligned} N(\mathbf{A}^0) \subset N(\mathbf{A}) \subset \cdots \subset N(\mathbf{A}^k) = N(\mathbf{A}^{k+1}) = N(\mathbf{A}^{k+2}) = \cdots \\ R(\mathbf{A}^0) \supset R(\mathbf{A}) \supset \cdots \supset R(\mathbf{A}^k) = R(\mathbf{A}^{k+1}) = R(\mathbf{A}^{k+2}) = \cdots \end{aligned} \quad (5.10.3)$$

*Proof.* If  $k \geq 0$  is the smallest integer such that  $R(\mathbf{A}^k) = R(\mathbf{A}^{k+1})$ , then  $R(\mathbf{A}^{i+k}) = R(\mathbf{A}^i \mathbf{A}^k) = \mathbf{A}^i R(\mathbf{A}^k) = \mathbf{A}^i R(\mathbf{A}^{k+1}) = R(\mathbf{A}^{i+k+1})$ . The rank plus nullity theorem (p. 199) insures that the nullspaces stop growing at the same place the ranges stop shrinking. ■

**Property 3.** *Let  $k$  be the value at which the ranges stop shrinking and the nullspaces stop growing in (5.10.3). For a singular  $\mathbf{A}_{n \times n}$  and an integer  $p > 0$ ,*

$$R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \mathbf{0} \iff R(\mathbf{A}^p) \oplus N(\mathbf{A}^p) = \mathfrak{R}^n \iff p \geq k.$$

*Proof.* If  $R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \mathbf{0}$ , use (4.4.19), (4.4.15), and (4.4.6), to write  $\dim[R(\mathbf{A}^p) + N(\mathbf{A}^p)] = \dim R(\mathbf{A}^p) + \dim N(\mathbf{A}^p) - \dim R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \dim R(\mathbf{A}^p) + \dim N(\mathbf{A}^p) = n \implies R(\mathbf{A}^p) + N(\mathbf{A}^p) = \mathfrak{R}^n$ .

Consequently,  $R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \mathbf{0}$  if and only if  $R(\mathbf{A}^p) \oplus N(\mathbf{A}^p) = \mathfrak{R}^n$ . Now prove  $R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \mathbf{0} \iff p \geq k$ . Suppose  $p \geq k$ . If  $\mathbf{x} \in R(\mathbf{A}^p) \cap N(\mathbf{A}^p)$ , then  $\mathbf{A}^p \mathbf{y} = \mathbf{x}$  for some  $\mathbf{y} \in \mathfrak{R}^n$ , and  $\mathbf{A}^p \mathbf{x} = \mathbf{0}$ , so  $\mathbf{A}^{2p} \mathbf{y} = \mathbf{A}^p \mathbf{x} = \mathbf{0} \implies \mathbf{y} \in N(\mathbf{A}^{2p}) = N(\mathbf{A}^p) \implies \mathbf{x} = \mathbf{0} \implies R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \mathbf{0}$ . Conversely, if  $R(\mathbf{A}^p) \cap N(\mathbf{A}^p) = \mathbf{0}$ , then  $R(\mathbf{A}^p) \oplus N(\mathbf{A}^p) = \mathfrak{R}^n$ , so

$$R(\mathbf{A}^p) = \mathbf{A}^p(\mathfrak{R}^n) = \mathbf{A}(R(\mathbf{A}^p)) = R(\mathbf{A}^{p+1}) \implies p \geq k. \quad \blacksquare$$

Below is a summary of our observations concerning the index of a matrix.

## Index

The index of a square matrix  $\mathbf{A}$  is the smallest nonnegative integer  $k$  such that any one of the three following statements is true.

- $\text{rank}(\mathbf{A}^k) = \text{rank}(\mathbf{A}^{k+1})$ .
- $R(\mathbf{A}^k) = R(\mathbf{A}^{k+1})$ —i.e., the point where  $R(\mathbf{A}^k)$  stops shrinking.
- $N(\mathbf{A}^k) = N(\mathbf{A}^{k+1})$ —i.e., the point where  $N(\mathbf{A}^k)$  stops growing.

For nonsingular matrices,  $\text{index}(\mathbf{A}) = 0$ . For singular matrices,  $\text{index}(\mathbf{A})$  is the smallest positive integer  $k$  such that either of the following two statements is true.

- $R(\mathbf{A}^k) \cap N(\mathbf{A}^k) = \mathbf{0}$ . (5.10.4)
- $\mathfrak{R}^n = R(\mathbf{A}^k) \oplus N(\mathbf{A}^k)$ .